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APPROXIMATE SOLUTION OF BELLMAN'S EQUATION FOR A CLASS OF OPTIMAL TERMINAL STATE CONTROL PROBLEMS

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We consider the problem of the optimal control of the terminal state of a linear system containing random perturbations in the form of Gaussian white noise. We propose a method for the approximate solution of Bellman's equation for one class of such systems in the case when the solution of the deterministic Bellman equation has discontinuities of the first kind in its values or in the values of its derivatives. As an application of the results obtained we give an approximate solution of Bellman's equation corresponding to one model problem in the control of entry into the atmosphere (see [1, 2]) and we compare the result obtained with the results of the numerical calculations in [2]. Some methods for the approximate solution of Bellman's equation have been studied earlier, for example, in [3 – 6]. Asymptotic expansions with respect to a small parameter, being the noise intensity, were constructed in [4 – 6] for the case when the deterministic Bellman equation corresponding to a system without random perturbations has a smooth solution. Exact solutions of Bellman's equation were obtained in [3] in certain cases when the system has a dimension of one.

1. Statement of the problem. Bellman's equation. Let the equation describing the motion of a system have the form

$$dx/dt = a(x, y, t) + b(x, y, t)u \tag{1.1}$$

Here $0 \leqslant t \leqslant T$, x is a scalar, u is the control function taking values in a convex closed set, $|u(t)| \leqslant p(t)$, $y = (y_1, \ldots, y_n)$ is a vector-valued function satisfying the equation

$$dy/dt = c(t) + d(t)y + \varepsilon e(x, y, t)\xi \tag{1.2}$$

Here $c(t)$, $d(t)$ are diagonal matrices of dimension n, $e(x, y, t)$ is a given matrix, ξ is the random perturbation vector, ε is a small parameter, $0 < \varepsilon < 1$. The functions a, b and the elements of matrices c, d, e are taken to be infinitely smooth functions of their arguments. The random perturbation vector is assumed to be a Gaussian white noise with unit intensity. The initial values $x(t_0) = x_0$, $y(t_0) = y_0$ being known, we are required to construct a control method minimizing (or maximizing) the mean of a scalar function $\psi[x(t)]$ of coordinate x at the terminal instant $t = T$. The function $\psi[x(t)]$ gives some measure of the deviation from a specified position at the end of the process.

If there are no random perturbations, we can find a solution $y(y_0, t)$ of system (1.2), and the system (1.1), (1.2) reduces to the scalar equation

$$dx/dt = a(x, y(y_0, t), t) + b(x, y(y_0, t), t)u \tag{1.3}$$

Problem (1.1), (1.2) models the motion of a controlled plant depending on the measurment of the n parameters $y_k(t)$, which can be deviated from certain prescribed values by random perturbations. Among problems of such kind we may include, for example, the problem of controlling a motion in a random medium.

Bellman's equation for problem (1.1), (1.2) has the form

$$S_\tau = \min_{|u| \leqslant p(\tau)} \{b(x, y, \tau)\ u S_x\} + a(x, y, \tau) S_x +$$

$$\sum_{k=1}^{n} (c_k(\tau) + d_k(\tau) y_k) S_{y_k} + \frac{\varepsilon^2}{2} \operatorname{Sp}(ee' S_{yy}) \tag{1.4}$$

with the initial condition

$$S(x, y, 0) = \psi(x) \tag{1.5}$$

Here $S(x, y, \tau)$ is Bellman's function, $T - t = \tau$ is the reverse time. The subscripts on the function S denote the taking of the corresponding partial derivatives, c_k and d_k are the diagonal elements of matrices c and d. Later on we shall use the notation

$$\operatorname{Sp}(ee' S_{yy}) = \sum_{i, j=1}^{n} f_{ij}(x, y, \tau) S_{y_i y_j}$$

reconing that the condition

$$\sum_{i, j=1}^{n} f_{ij}\xi_i\xi_j > 0, \qquad \xi \neq 0 \tag{1.6}$$

is fulfilled for any real vector $\xi = (\xi_1, \ldots \xi_n)$.

We introduce the new variable

$$z_k = y_k \exp\left[\int_{\tau_0}^{\tau} d_k(\lambda)\, d\lambda\right] - \int_{\tau_0}^{\tau} c_k(\lambda) \exp\left[\int_{\tau_0}^{\lambda} d_k(\lambda_1)\, d\lambda_1\right] d\lambda \tag{1.7}$$

$$\tau_0 = T - t_0, \quad k = 1, \ldots, n$$

By computing in (1.4) the operation of taking the minimum, we obtain

$$\min_{|u| \leqslant p(\tau)} \{b(x, y, \tau) u S_x\} = -p(\tau) | b(x, y, \tau) S_x | = F(x, y, \tau, S_x)$$

$$u = \begin{cases} p(\tau), & \operatorname{sign}(bS_x) < 0 \\ -p(\tau), & \operatorname{sign}(bS_x) \geqslant 0 \end{cases} \tag{1.8}$$

With due regard to (1.7),(1.8) the boundary value problem (1.4),(1.5) takes the form

$$S_\tau = -p(\tau)\,|b^1 S_x| + a^1 S_x + \frac{1}{2}\varepsilon^2 \sum_{i,j=1}^{n} f^1_{i,j} S_{z_i z_j}$$
$$S(x, z, 0) = \psi(x) \tag{1.9}$$

Here S is Bellman's function of the variables (x, z, τ), defined in the region

$$\Omega = \{x, z, \tau : -\infty \leqslant x \leqslant +\infty, \quad -\infty \leqslant z_k \leqslant +\infty, k = 1, \ldots, n, \quad 0 \leqslant \tau \leqslant T\}.$$

The functions obtained from the functions a, b, f_{ij} by the change of variables (1.7) have been denoted by a^1, b^1, $f_{ij}{}^1$, respectively.

Note 1. The subsequent discussions remain valid also in the case when Eq. (1.1) additively contains a random perturbation in the form of Gaussian white noise of unit intensity with a small parameter ε, i.e.

$$dx/dt = a(x, y, t) + b(x, y, t)u + \varepsilon e_1(x, y, t)\xi_1$$

Here ξ_1 is Gaussian white noise independent of ξ, $e_1(x, y, t)$ is some infinitely smooth function of its arguments.

Note 2. An existence and uniqueness theorem for the solution of problem (1.1), (1.2) was proven in [7].

2. Solution of the deterministic problem. Determination of the characteristics of the deterministic equation.

We assume that we know the Bellman function S° corresponding to the deterministic problem (1.3), and

$$S^\circ(x, z, \tau) = \begin{cases} S^{\circ 1}(x, z, \tau), & A^\circ(x, z, \tau) \leqslant 0 \\ S^{\circ 2}(x, z, \tau), & A^\circ(x, z, \tau) > 0 \end{cases} \tag{2.1}$$

Here $A^\circ(x, z, \tau)$ is some continuous function of its arguments. The functions $S^{\circ k}$, $k = 1, 2$, continuous outside the set A°, are constructed such that the function S° is subject to discontinuities of the first kind in its own values or in the values of its derivatives on the surface $A^\circ = 0$. It is clear that such a case corresponds to a discontinuous function $\psi(x)$ of initial values. Further, we assume that the function S° is defined for all values of x. Bellman's equation corresponding to the deterministic problem has the form

$$S_\tau^\circ = -p(\tau)\,|b^1 S_x^\circ| + a^1 S_x^\circ \tag{2.2}$$
$$S^\circ(x, z, 0) = \psi(x) \tag{2.3}$$

From the theory of partial differential equations we know that in equations of hyperbolic type the discontinuities of the initial values and of its derivatives are spread over the characteristics, therefore, it is natural to expect that in spite of the nonlinearity the surface $A^\circ = 0$ is the characteristic surface for Eq. (2.2). However, in the case of Eq. (2.2) the concept of a characteristic surface itself is needed in the definition. Indeed, in this case the equation of the characteristic surface depends upon the sign of the expression (bS_x°) and, by the same token, on the solution of Eq. (2.2) itself, but since S° can be a step function (see [1, 2]) taking the value zero where $A^\circ \leqslant 0$ and unity when

$A^\circ > 0$, it is not clear in this case what sign (bS_x°) means. Therefore, we carry out the following additional constructions.

We consider the fundamental solution of the equation $p_\tau^\mu = {}^1/_2\ \mu^2\ p_{xx}^\mu$, $\mu > 0$, having the form

$$p^\mu(x, \tau) = [2\mu\sqrt{\pi\tau}\,]^{-1}\exp[-x^2/4\mu^2\tau] \tag{2.4}$$

and we construct the function

$$S^\mu(x, z, \tau) = \int_{-\infty}^{+\infty} S^\circ(\lambda, z, \tau)\, p^\mu(x-\lambda, \tau)\, d\lambda$$

Here $S^\circ(\lambda, z, \tau)$ is the function defined by equality (2.1) and is a solution of deterministic Eq. (2.2). From the properties of the fundamental solution it follows [8] that

a) S^μ is infinitely differentiable with respect to x in Ω;

b) $S^\mu \to S^\circ$, $S_x^\mu \to S_x^\circ$ as $\mu \to 0$ in the sense of convergence in the space of summable functions;

c) $S^\mu(x, z, 0) = S^\circ(x, z, 0) = \psi(x)$

Let us consider the function $F(x, z, \tau, S_x^\mu)$ defined by equality (1.8). For every fixed $\mu > 0$ the function bS_x^μ takes nonnegative values on a certain set and strictly negative ones on the complement of this set in Ω. Thus, for each μ the region Ω is divided into the regions $\Omega_\mu^+ = \{x, z, \tau\colon b^1S_x^\mu \geqslant 0\}$ and $\Omega_\mu^- = \{x, z, \tau\colon b^1S_x^\mu < 0\}$. We assume that in Ω_μ^+ Eq. (2.2) has the form

$$S_\tau^\circ = -p(\tau)\, b^1S_x^\circ + a^1S_x^\circ$$

and in region Ω_μ^-, the form

$$S_\tau^\circ = p(\tau)\, b^1S_x^\circ + a^1S_x^\circ$$

In each of the regions Ω_μ^+ and Ω_μ^- these equations have families of characteristic surfaces which are given by the equations $\eta = A_\mu^+(x, z, \tau)$, $\eta = A_\mu^-(x, z, \tau)$, $\eta =$ const, A_μ^+ and A_μ^- are some smooth functions of their arguments. From continuity considerations it is clear that the characteristic surfaces corresponding to the regions Ω_μ^+ and Ω_μ^- are matched in continuous fashion along the boundary of the region Ω_μ^+ for each corresponding value of η. A family of surfaces $A_\mu(x, z, \tau) = \eta$, $\eta =$ const, which we call the μ-characteristic family of surfaces of Eq. (2.2), is formed for each μ.

Assumption 1. There exists a value μ^* such that the μ-characteristic families of Eq. (2.2) coincide identically for $0 < \mu < \mu^*$, i.e. $A_\mu(x, z, \tau) = A(x, z, \tau)$, $0 < \mu < \mu^*$.

Definition 1. The family of surfaces $\eta = A(x, z, \tau)$, $\eta =$ const, is said to be characteristic for Eq. (2.2).

Assumption 2. The surface $A^\circ(x, z, \tau) = 0$, appearing in the determination of the solution (2.1) of the deterministic problem (1.4), is characteristic in the sense of Definition 1.

Without loss of generality we can assume that surface $A^\circ(x, z, \tau) = 0$ corresponds to the constant value $\eta = 0$, i.e. $0 = \eta = A(x, z, \tau) = A^\circ(x, z, \tau)$, and that the values of the constant η defining the characteristic surfaces of Eq. (2.2) vary within the limits $-\infty \leqslant \eta_1 \leqslant \eta \leqslant \eta_2 \leqslant +\infty$. From (2.1) it follows that the solution of deterministic problem (1.3) can be written in the form

$$S^\circ(x, z, \tau) = S^\circ(\eta) = \begin{cases} S^{\circ 1}(\eta), & \eta \leqslant 0 \\ S^{\circ 2}(\eta), & \eta > 0 \end{cases} \tag{2.5}$$

Hence follows

$$S^\circ(x, z, 0) = S^\circ(\eta)|_{\tau=0} = \psi(x)$$

Assumption 3. The functions $S^{\circ k}$, $k = 1, 2$ are such that

a) $S_\eta^\circ(\eta) > 0, \qquad \eta \neq 0$

b) $S_\eta^\circ(+0) - S_\eta^\circ(-0) \geqslant 0.$

Here the subscript η denotes the taking of the derivative of the function $S^\circ, S_\eta^\circ(\pm 0) = \lim S_\eta^\circ$ as $\eta \to \pm 0$, respectively.

Note 3. If in the original statement of problem (1.1), (1.2) we look for not the minimum but the maximum of the function $\psi[x(T)]$, then conditions (a) and (b) in Assumption 3 take the form

a') $S_\eta^\circ(\eta) \leqslant 0, \qquad \eta \neq 0$

b') $S_\eta^\circ(+0) - S_\eta^\circ(-0) \leqslant 0.$

From the definition of the characteristic surfaces of Eq. (2.2) we see that in them there can be conic points when (x, z, τ) is such that $b(x, z, \tau)\ S_x^\mu = 0$, $0 < \mu < \mu^*$. By construction this set is a limit set for the set Ω_μ^-.

Definition 2. The limit of the values of the derivatives of the corresponding variables as the point (x, z, τ) belonging to region Ω^+ tends to the point (x^*, z^*, τ^*), is called the derivative of the function $\eta = A(x, z, \tau)$, being a characteristic surface of Eq. (2.2) at the conic point (x^*, z^*, τ^*).

Assumption 4. The condition

$$\frac{\partial\eta}{\partial x} = \frac{\partial A(x, z, \tau)}{\partial x_j} \neq 0 \qquad (2.6)$$

is valid for all $(x, z, \tau) \in \Omega$.

Let us consider an example illustrating the assumptions and definitions we have introduced. Let $dx/dt = u + \xi$, x is a scalar, $0 \leqslant t \leqslant T$, $\xi(t)$ is Gaussian white noise, $|u(t)| \leqslant p_0 = \text{const}$. The function $\psi[x(T)]$ is given by the equality

$$\psi(x) = \begin{cases} 0, & |\ \ | \leqslant l_0 \\ 1, & |x| > l_0 \end{cases}$$

Bellman's equation has the form

$$S_\tau = -p_0|S_x| + {}^1/_2 S_{xx}, \qquad S(x, 0) = \psi(x)$$

The function S°, being the solution of the deterministic problem, has the form (see [3])

$$S^\circ(x, \tau) = \begin{cases} 0, & |x| \leqslant l_0 + p_0\tau \\ 1, & |x| > l_0 + p_0\tau \end{cases}$$

We write the Bellman equation corresponding to the deterministic problem

$$S_\tau^\circ = -p_0|S_x^\circ|, \qquad S^\circ(x, 0) = \psi(x) \qquad (2.7)$$

Having set up the function S^μ appearing in the construction of the μ-characteristic surfaces of the deterministic equation, we have

$$S^\mu = 1 - \int\limits_{|\lambda| \leqslant l_0 + p_0\tau} p^\mu(x - \lambda, \tau)\, d\lambda$$

Here $p^\mu(x, \tau)$ is the fundamental solution of the equation $S_\tau^\mu = {}^1/_2\mu^2 S_{xx}^\mu$, defined by formula (2.4).

We consider the regions $\Omega_\mu^+ = \{x, \tau : S_x^\mu \geqslant 0\}$ and $\Omega_\mu^- = \{x, \tau: \ S_x^\mu < 0\}$. Differentiating S^μ with respect to x, we obtain

$$S_x{}^\mu = [\mu \sqrt{2\pi\tau}]^{-1} \exp\left\{-\frac{[x-(l_0+p_0\tau)]^2}{4\mu^2\tau}\right\}\left\{1-\exp\left[\frac{-4x(l_0+p_0\tau)}{4\mu^2\tau}\right]\right\}$$

Hence we see that for all $\mu > 0$ the sets Ω_μ^+ coincide with the halfplane $x \geqslant 0$ and the set Ω_μ^- with the halfplane $x < 0$. Consequently, the conditions of Assumption 1 are fulfilled and the μ-characteristic Eqs. (2.7) coincide identically for all $\mu \geqslant 0$ and are given by the expressions $\eta = x - p_0\tau$ when $x \geqslant 0$ and $\eta = -(x + p_0\tau)$ when $x < 0$, $\eta = \text{const}$. By virtue of Definition 1 these straight lines are the characteristic Eqs. (2.7). The staight lines $l_0 = x - p_0\tau$ and $l_0 = -(x + p_0\tau)$, defining the lines of discontinuity of the function S°, are the characteristics of the deterministic Bellman equation (2.7), originating at the points $(x = l_0, \tau = 0)$ and $(x = -l_0, \tau = 0)$, and thus, Assumption 2 is fulfilled. It is easy to verify that the conditions of Assumption 3 are fulfilled. From Definition 2 it follows that $\partial\eta / \partial x = 1$ when $x \geqslant 0$ and $\partial\eta / \partial x = -1$ when $x < 0$, i.e. Assumption 4 is fulfilled.

3. Construction of the approximate solution. For any point (x, z, τ) we seek the solution of the boundary value problem (1.9), (1.10) as a function of the values of the constant η such that $\eta = A(x, z, \tau)$ and of the values of z, τ. By virtue of Assumption 4 and by the implicit function theorem it follows that $x = s(\eta, z, \tau)$; therefore, the solution of problem (1.9) can be treated as a function of the variables (η, z, τ). We denote this function by $S^\varepsilon(\eta, z, \tau)$. We set $\eta_1 = \eta / \varepsilon$. The following relations are valid:

$$S_x{}^\varepsilon = \frac{1}{\varepsilon} S^\varepsilon_{\eta_1} \frac{\partial\eta}{\partial x}, \qquad S^\varepsilon_{z_i} = \frac{1}{\varepsilon} S^\varepsilon_{\eta_1}\frac{\partial\eta}{\partial z_i} + S^\varepsilon_{z_i}, \qquad i = 1, \ldots, n$$

$$S_\tau{}^\varepsilon = S_\tau{}^\varepsilon| + \frac{1}{\varepsilon} S^\varepsilon_{\eta_1}\frac{\partial\eta}{\partial\tau}, \qquad S^\varepsilon_{z_iz_j} = \frac{1}{\varepsilon^2} S^\varepsilon_{\eta_1\eta_1}\frac{\partial\eta}{\partial z_i}\frac{\partial\eta}{\partial z_j} +$$

$$\frac{1}{\varepsilon} S^\varepsilon_{\eta_1}\frac{\partial^2\eta}{\partial z_i\partial z_j} + \frac{1}{\varepsilon} S^\varepsilon_{z_i\eta_1}\frac{\partial\eta}{\partial z_j} + \frac{1}{\varepsilon} S^\varepsilon_{\eta_1z_j}\frac{\partial\eta}{\partial z_i} + S^\varepsilon_{z_iz_j}$$

$$i, j = 1, \ldots n$$

Taking into account equality (2.5) which defines the solution of the deterministic problem, we can assume that

$$S^\varepsilon_{\eta_1}(\eta_1, z, \tau) \geqslant 0, \qquad (x, z, \tau) \in \Omega \tag{3.1}$$

Therefore, in order that the desired function S satisfy Eq. (1.9) it is necessary that

$$S_\tau{}^\varepsilon = -\frac{1}{\varepsilon} S^\varepsilon_{\eta_1}\left[\frac{\partial\eta}{\partial\tau} + p(\tau)\left|b^1\frac{\partial\eta}{\partial x}\right| - a^1\frac{\partial\eta}{\partial x}\right] + \frac{\varepsilon^2}{2}\sum_{i,j=1}^n f_{ij}{}^1 \times$$

$$\left[\frac{1}{\varepsilon^2} S^\varepsilon_{\eta_1\eta_1}\frac{\partial\eta}{\partial z_j}\frac{\partial\eta}{\partial z_i} + \frac{1}{\varepsilon} S^\varepsilon_{\eta_1}\frac{\partial^2\eta}{\partial z_i\partial z_j} + \frac{1}{\varepsilon} S^\varepsilon_{z_i\eta_1}\frac{\partial\eta}{\partial z_j} + \frac{1}{\varepsilon} S^\varepsilon_{\eta_1z_j}\frac{\partial\eta}{\partial z_i} + S^\varepsilon_{z_iz_j}\right]$$

Here the functions obtained under the change of variables $x = s(\eta, z, \tau)$ have been denoted by the same letters.

Since $\eta = A(x, z, \tau)$ are the characteristics of the deterministic Bellman equation (2.2),

$$\frac{\partial\eta}{\partial\tau} = -p(\tau)\left|b^1\frac{\partial\eta}{\partial x}\right| + a^1\frac{\partial\eta}{\partial x}$$

As a result we obtain that the function S^ε should satisfy the boundary value problem

$$S_\tau{}^\varepsilon = \frac{1}{2} E_1 S^\varepsilon_{\eta_1\eta_1} + \frac{\varepsilon}{2} E_2 S^\varepsilon_{\eta_1} + \frac{\varepsilon}{2} \sum_{i,j=1}^{n} f_{ij}{}^1 \frac{\partial\eta}{\partial z_j} S^\varepsilon_{\eta_1 z_i} + \frac{\varepsilon^2}{2} \sum_{i,j=1}^{n} f_{ij}{}^1 S^\varepsilon_{z_i z_j}, \qquad S^\varepsilon(\eta_1, z, \tau)|_{\tau=0} = \psi(x) \tag{3.2}$$

Here

$$E_1 = \sum_{i,j=1}^{n} f_{ij}{}^1 \frac{\partial\eta}{\partial z_i \partial z_j}, \qquad E_2 = \sum_{i,j=1}^{n} f_{ij}{}^1 \frac{\partial^2\eta}{\partial z_i \partial z_j}$$

We seek the solution of problem (3.2) in the form of a power series

$$S^\varepsilon(\eta_1, z, \tau) = S^1(\eta_1, z, \tau) + \varepsilon S^2(\eta_1, z, \tau) + O(\varepsilon^2) \tag{3.3}$$

Substituting the function S^ε, represented in form (3.3), into Eq. (3.2), we obtain that the function S^1 should satisfy the boundary value problem

$$S_\tau{}^1 = \frac{1}{2} E_1 S^1_{\eta_1\eta_1}, \qquad S^1(\eta_1, z, \tau)|_{\tau=0} = \psi(x) \tag{3.4}$$

and the condition

$$S^1_{\eta_1} \geqslant 0 \tag{3.5}$$

The function S^2 should be chosen such that

$$S_\tau{}^2 = \frac{1}{2} E_1 S^2_{\eta_1\eta_1} + G(\eta_1, z, \tau, S^1), \qquad S^2(\eta_1, z, \tau)|_{\tau=0} = 0$$

$$G(\eta_1, z, \tau, S^1) = \sum_{i,j=1}^{n} f_{ij}{}^1 S^1_{\eta_1 z_i} + E_2 S^1_{\eta_1} \tag{3.6}$$

Let us assume that the functions S^1 and S^2, satisfying the boundary value problems (3.4) and (3.6), respectively, have been found. We consider the function $W = S^1 + \varepsilon S^2$. The following assertion is valid.

Theorem. Let condition (1.6) be fulfilled and let the function S^ε be a solution of problem (3.2), satisfying condition (3.1). Then

$$|S^\varepsilon - W| = O(\varepsilon^2)$$

The proof relies on the following lemma which follows directly (after the substitution $T - t = \tau$) from Theorem 10 in [9] (see p. 16).

Lemma. Let a continuous and bounded function be a solution of the following Cauchy problem:

$$\sum_{i,j=1}^{n} \alpha_{ij}(\tau, x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \beta_i(\tau, x) \frac{\partial v}{\partial x_i} + \gamma(\tau, x) = \delta(\tau, x) + \frac{\partial v}{\partial \tau}$$

where the matrix α_{ij} is positive definite for all $\tau \in [0, T]$, $x \in R^n$. We have

$$|v_0(x)| \leqslant K_1, \; |\delta(\tau, x)| \leqslant K_2, \; |\gamma(\tau, x)| \leqslant K_3,$$

and the coefficients α_{ij}, β_i $(i, j = 1, \ldots, n)$ satisfy the conditions

$$|\alpha_{ij}(\tau, x)| \leqslant K_4 (\|x\|^2 + 1), \; |\beta_i(\tau, x)| \leqslant K_5 (\|x\| + 1)^{1/2}$$

Here $K_1, \ldots, K_5$ are nonnegative constants. Under these assumptions

$$|v(\tau, x)| \leqslant (K_1 + K_2\tau) \exp[K_3\tau], \quad 0 \leqslant \tau \leqslant T, \; x \in R^n$$

Indeed, we obtain the required estimate immediately from the lemma by substituting the function $S^\varepsilon - W$ into Eq. (3.2) and by using the fact that

$$(S^\varepsilon - W)|_{\tau=0} = 0$$

Let us consider certain possible cases of the realization of boundary value problems (3.4) and (3.6).

1°. The functions E_1 and E_2 are functions of the valiables z and τ, the values of $\eta \in (-\infty, +\infty)$. Boundary value problem (3.4), in this case, takes the form

$$S_\tau^1 = \frac{1}{2} E_1(z,\tau) S_{\eta_1\eta_1}, \quad S^1(\eta_1, z, \tau)|_{\tau=0} = \psi(x) \tag{3.7}$$

The fundamental solution of the problem is determined by the expression

$$p(\eta_1, z, \tau) = [2\sqrt{\pi E_1'(z,\tau)}]^{-1} \exp\left[-\frac{\eta_1^2}{4E_1'(z,\tau)}\right]$$

$$E_1'(z,\tau) = \int_0^\tau E_1(z,\lambda)\, d\lambda$$

Let $S^\circ(\eta)$ be the solution of the corresponding deterministic Bellman equation. Then the solution of boundary value problem (3.7) is given by the formula

$$S^1(\eta_1, z, \tau) = \int_{-\infty}^{+\infty} S^\circ(\lambda)\, p(\eta_1 - \lambda, z, \tau)\, d\lambda \tag{3.8}$$

Indeed, by a direct check we can be convinced that the function S^1, constructed from formula (3.8), is a solution of Eq. (3.7). From the properties of a fundamental solution it follows that boundary condition (3.7) is fulfilled. Let us verify the fulfillment of condition (3.5)

$$S_{\eta_1}^1 = -\int_{-\infty}^{+\infty} \left[\frac{\partial}{\partial\lambda} p(\eta_1 - \lambda, z, \tau)\right] S^\circ(\lambda)\, d\lambda = \int_{-\infty}^{+\infty} \left[\frac{\partial}{\partial\lambda} S^\circ(\lambda)\right] p(\eta_1 - \lambda, z, \tau)\, d\lambda$$

Here the derivative of function S° is understood in the generalized sense. By virtue of the conditions in Assumption 3 we obtain that $S_{\eta_1}^1 \geqslant 0$.

Boundary value problem (3.6) takes the form

$$S_\tau^2 = {}^1/_2 E_1(z,\tau) S^2_{\eta_1\eta_1} + G(z, \tau, S^1), \qquad S^2(\eta_1, z, \tau)|_{\tau=0} = 0$$

The solution of this problem is given by the formula (see [10], for example)

$$S^2(\eta_1, z, \tau) = \int_0^\tau \int_{-\infty}^{+\infty} G(\lambda, z, \lambda_1, S^1(\lambda, z, \lambda_1))\, p(\eta_1 - \lambda, z, \tau - \lambda_1)\, d\lambda\, d\lambda_1$$

If the values of the constant η defining the families of characteristics of Eq. (2.2) vary within the following intervals:

$$-\infty < \eta_1 < \eta \leqslant +\infty, \quad -\infty < \eta \leqslant \eta_2 < +\infty, \quad -\infty < \eta_1 \leqslant \eta \leqslant \eta_2 < +\infty$$

then in each of these cases it is necessary to specify the boundary conditions at the end points $\eta = \eta_1$, $\eta = \eta_2$ and $\eta = \eta_1$, $\eta = \eta_2$, respectively. Here can arise the first, second or third boundary value problem for the parabolic equation (3.4). Methods for solving such problems have been worked out very well and have been set forth, for example, in monographs [8, 10].

2°. The functions E_1 and E_2 depend upon the variable η_1. In this case the method for constructing the fundamental solution, proposed by Levy (see [10]), can be applied to solve the boundary value problems (3.4) and (3.6). As a result the boundary value problem reduces to the solving of an integral equation of the second kind which can be obtained by the method of successive approximations.

Note 4. If in the original statement of problem (1.1),(1.2) we look for the maximum of the function $\psi[x(T)]$, then in addition to conditions (a'),(b') of Assumption 3, condition (3.1) takes the form

$$S_{\eta_1}{}^{\varepsilon}(\eta_1, z, \tau) \leqslant 0, \qquad (x, z, \tau) \in \Omega$$

4. Approximate solution of Bellman's equation for a model optimal control problem of entry into the atmosphere. Let the equation of motion of a material point have the form

$$d^2x / dt^2 = u(t)\,\xi(t) \tag{4.1}$$

Here $0 \leqslant t \leqslant T$, $u(t)$ is the control function, x is a scalar, $|u(t)| \leqslant u_0$, $\xi(t)$ is a random function representing a stationary Gaussian process with unit mean and correlation function

$$M\{[\xi(t)-1][\xi(t+\tau)-1]\} = \sigma^2 e^{-k\tau}, \quad \tau > 0$$

characterizing the behavior of a random medium in which the motion takes place. The initial values $x(0)$ and $x^{\cdot}(0)$ being known, we are required to construct a control method which would maximize the probability of falling into the region $[-\delta, +\delta]$, $\delta > 0$ on the x-axis at a terminal instant T. It is assumed that $\xi(t)$ is a Markov process.

Then the control, optimal in the sense indicated, depends on x, $x^{\cdot}$ and ξ. We set $x_1 = x + x^{\cdot}(T - t)$, $y = \xi$. Equation (4.1) is written as

$$x_1^{\cdot} = (T-t)\,yu, \quad y^{\cdot} = -k(y-1) + \sigma\sqrt{2k}\,\xi_1$$

Here ξ_1 is Gaussian white noise. The last equation must be understood in Ito's sense. Bellman's equation and the initial condition in this case have the form [1, 2]

$$S_\tau = \max_{|u|\leqslant u_0}\{\tau y u S_{x_1}\} - k(y-1)S_y + \frac{k\sigma^2}{2}S_{yy} \tag{4.2}$$

$$S(x_1, y, 0) = \psi(x_1) = \begin{cases}1, & |x_1| \leqslant \delta\\ 0, & |x_1| > \delta\end{cases} \tag{4.3}$$

Here $T - t = \tau$ is reverse time. Computing the maximum in (4.2), we obtain

$$u = u_0 \operatorname{sign}(yS_{x_1}), \quad \max_{|u|\leqslant u_0}\{\tau y u S_{x_1}\} = u_0\tau\,|yS_{x_1}|$$

We introduce the new variable $y_1 = (y-1)e^{k\tau}$. In the region

$$\Omega = \{x_1, y_1, \tau: -\infty < x_1 < +\infty, -\infty < y_1 < +\infty, 0 < \tau \leqslant T\}$$

the boundary value problem (4.2),(4.3) takes the form

$$S_\tau = u_0\tau\,|(1 + y_1 e^{-k\tau})S_{x_1}| + \frac{\sigma^2 k}{2}e^{2k\tau}S_{y_1y_1}$$

$$S(x_1, y_1, 0) = \psi(x) \tag{4.4}$$

In the deterministic case the quantity y, modelling the fluctuations in the atmosphere, is a constant and Bellman's equation is written as

$$S_\tau{}^\circ = u_0\tau\,|\,yS_{x_1}{}^\circ\,|, \qquad S^\circ(x_1, y, 0) = \psi(x_1) \tag{4.5}$$

Here S° is Bellman's function corresponding to deterministic problem (4.1). The solution of the deterministic problem is given by the formula

$$S^\circ = \begin{cases} 1, & |x_1| \leqslant \delta + {}^1/_2 u_0\tau^2\,|\,y\,| \\ 0, & |x_1| > \delta + {}^1/_2 u_0\tau^2\,|\,y\,| \end{cases} \tag{4.6}$$

Formula (4.6) reflects the fact that the region, from which we can fall into the interval $[-\delta, \delta]$ by the instant $t = T$, contracts down to the very interval $[-\delta, \delta]$ as the time $t \leqslant T$ increases (i.e. as the value of $\tau = T - t$ decreases); the boundary of this region for each fixed value of y is given by the equation $|x_1| = \delta + {}^1/_2 u_0\tau^2\,|\,y\,|$ being, as is shown later, the equation for the characteristics originating at the points $(x_1 = \delta,\ \tau = 0)$, $(x_1 = -\delta,\ \tau = 0)$.

In accordance with Sect. 2 we consider the fundamental solution of the boundary value problem

$$p_\tau{}^\mu = {}^1/_2\mu^2 p^\mu_{x_1x_1}$$

and we construct the function

$$S^\mu(x_1, y, \tau) = \int_{-\infty}^{+\infty} S^\circ(\lambda, y, \tau)\, p^\mu(x_1 - \lambda, \tau)\, d\lambda \tag{4.7}$$

Here $p^\mu(x_1, \tau)$ is defined by formula (2.4). Using (4.6) the last integral can be written as

$$S^\mu = [2\mu\sqrt{\pi\tau}]^{-1}\int_a^b \exp\left[-\frac{(x_1-\lambda)^2}{4\mu^2\tau}\right] d\lambda,\ a = \delta + \frac{1}{2}u_0\tau^2\,|\,y\,|,\ b = -\delta - \frac{1}{2}u_0\tau^2\,|\,y\,|$$

Therefore,

$$S_{x_1}{}^\mu = [2\mu\sqrt{\pi\tau}]^{-1}\exp\left[-\frac{(x_1+a)^2}{4\mu^2\tau}\right]\left[1 - \exp\frac{4x_1a}{4\mu^2\tau}\right]$$

Hence we see that for all $\mu > 0$, $S^\mu_{x_1} \leqslant 0$, when $x_1 \geqslant 0$ and $S^\mu_{x_1} > 0$, when $x_1 < 0$. Consequently, the requirements of Assumption 1 are valid and, by virtue of Definition 1, the characteristics of Eq. (4.5) are given by the equations $\eta = |\,x_1\,| - {}^1/_2\,u_0\tau^2\,|\,y\,|$, $\eta = \text{const}$. The solution of Eq. (4.5) defined by equality (4.6) is written in the following form:

$$S^\circ(\eta) = \begin{cases} 1, & \eta \leqslant \delta \\ 0 & \eta > \delta \end{cases}$$

and thus, the conditions of Assumption 2 are fulfilled. Further, it is easy to verify that conditions (a') and (b') of Assumption 3 and condition (2.6) of Assumption 4 are filfilled.

In accordance with Sect. 3 we seek the solution of problem (4.4) as a function $S^\sigma(\eta, y_1, \tau)$ of values of η such that $\eta = |\,x_1\,| - {}^1/_2 u_0\tau^2\,|\,y\,|$ of the quantities y_1, τ. Here $y = 1 + y_1e^{-k\tau}$ is a variable quantity; we assume that the fixed quantity σ is sufficiently small. We set $\eta_1 = \eta\,/\,\sigma$ and we seek the solution as an asymptotic series in powers of σ

$$S^\sigma = S^1(\eta_1, y_1, \tau) + \sigma S^2(\eta_1, y_1, \tau) + O(\sigma^2)$$

By virtue of Definition 2

$$\frac{\partial\eta}{\partial x_1} = \begin{cases} 1, & x_1 \geqslant 0 \\ -1, & x_1 < 0 \end{cases}, \quad \frac{\partial\eta}{\partial y_1} = \frac{1}{2}u_0\tau^2 e^{-k\tau}\,\mathrm{sign}\,(1 + y_1e^{-k\tau})$$

From (3.4) it follows that the function S^1 should satisfy the boundary value problem

$$S_\tau^1 = \frac{1}{2} k \left(\frac{1}{2} u_0 \tau^2\right)^2 S^1_{\eta_1\eta_1}, \qquad S^1(\eta_1, y_1, \tau)\,|_{\tau=0} = \psi(\eta_1)\,|_{\tau=0} \tag{4.8}$$

The condition

$$S_{\eta_1}^1(\eta_1, y_1, \tau) \leqslant 0 \tag{4.9}$$

should be fulfilled by virtue of Note 4. Boundary value problem (4.8) is to be solved in explicit form. To do this it is necessary to write down the fundamental solution of problem (4.8) and to examine the convolution of this solution with the solution of the deterministic Bellman equation just as was done in Case 1 in Sect. 3. As the result we obtain

$$S^1(\eta_1, \tau) = \left[2\sqrt{\frac{\pi u_0^2 k}{20}\tau^5}\right]^{-1} \int_{-\infty}^{+\infty} S^\circ(\lambda) \exp\left[-\frac{(\eta_1-\lambda)^2 20}{4u_0^2 k \tau^5}\right] d\lambda =$$

$$[2\sqrt{\pi M}\,]^{-1} \int_{-\infty}^{\delta} \exp\left[-\frac{(\eta_1-\lambda)^2}{4M}\right] d\lambda, \quad M = \frac{u_0^2 k}{20}\tau^5$$

By introducing the new variable z, we obtain

$$S^1(\eta_1, \tau) = \frac{1}{\sqrt{\pi}} \int_{z_1}^{+\infty} e^{-z^2} dz = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2} dz, \quad z_1 = \frac{\eta_1 - \delta}{2\sqrt{M}} \tag{4.10}$$

By direct verification we can be convinced that the function S^1 is the solution of boundary value problem (4.8) and satisfies condition (4.9). The function S^2 must be chosen such that

$$S_\tau^2 = \frac{1}{2} k \left(\frac{1}{2} u_0 \tau^2\right)^2 S^2_{\eta_1\eta_1}, \quad S^2_{(\eta_1, y_1, \tau)}\,|_{\tau=0} = 0$$

From the uniqueness of the solution of the Cauchy problem for the heat conduction equation it follows that $S^2 = 0$. By virtue of the theorem in Sect. 3, formula (4.10) gives the approximate solution of problem (4.4), (4.5), differing from the exact one by a quantity of the order $O(\sigma^2)$. Figure 1 shows a comparison of the solution obtained by the approximate formula (4.10) and of the results of numerical calculations in [2]. The solid line depicts the curves obtained by numerical calculation. The curves *1* and *2* correspond to the values $y = 0.15$ and 1.8 with $\tau = 1.4$, $\sigma = 0.25$, $\delta = 1$, $k = 1$, $u_0 = 1$. The dotted lines are obtained by using the approximate formula (4.10) and correspond to the same values of the parameters y, τ, σ, k, u_0, as for the curves obtained numerically.

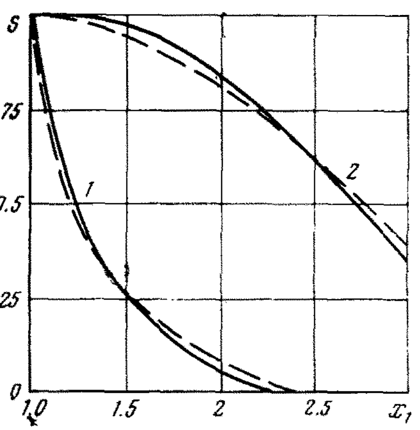

Fig. 1

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ON MINIMAL OBSERVATIONS IN A GAME OF ENCOUNTER

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We consider the differential game of the encounter of "isotropic rockets" [1]. Its solution, under the condition of complete informativeness of the players, has been constructed in [2]. We investigate the question of the minimal information needed by the players to realize a saddle situation. The statement of similar game problems with incomplete information has been given in [3].

1. Let the motion of players X and Y on a fixed time interval $[0, T]$, $T > 0$ be specified by the relations

$$X: x_1^{\cdot} = x_2, \quad x_2^{\cdot} = u, \quad |u| \leqslant 1, \qquad Y: y^{\cdot} = v, \quad |v| \leqslant 1$$
$$x_1(0) = x_1^{\circ}, \quad x_2(0) = x_2^{\circ}; \qquad y(0) = y^{\circ} \tag{1.1}$$

Here x_1, x_2, u, y, v are vectors of arbitrary like dimension. Player X has the following information available to him. At each instant $t \in [0, T]$ he knows the exact value of the natural phase coordinate vectors $x_1(t)$, $x_2(t)$. Player X observes the opponent's